Journal of Engineering Physics and Thermophysics, Vol. 73, No. 5, 2000

DESCRIPTION OF A SET OF SOLUTIONS OF A DISSIPATION INEQUALITY FOR THIRD-ORDER RELAXATION SYSTEMS

V. T. Borukhov and D. M. Zelenyak

UDC 517.938+536.7

For the class of linear dynamic systems indicated in the title, a variety of the third degree is determined and it is proved that the set of solutions of a dissipation inequality is characterized by a limited convex component of this variety. Singular varieties are determined, and the relationship between them and the maximum and minimum elements of the set of solutions for the dissipation inequality is established. The results obtained can find application in the theory of linear viscoelasticity and the theory of generalized thermodynamic systems with memory.

The applications of the theory of passive and relaxation dynamic systems are closely connected with the problem of mathematical simulation of the response of physical objects to an external action. In nonequilibrium thermodynamics the conditions of passivity and also narrower conditions of the relaxation state make it possible to isolate a thermodynamically narrow class of governing equations [1-5] required to close balance equations.

Investigation of linear passive systems in the phase space of states is based on a dissipation inequality, defined in [6, 7], in connection with the problem of absolute stability of nonlinear systems of differential equations. On the other hand, dissipation inequalities understood in a broader sense were always important for representing the second law of thermodynamics in one form or another [1, 2, 8]. In [3-5] it is shown that sets of solutions of dissipation inequalities for linear passive systems determine the classes of physically admissible nonequilibrium thermodynamic potentials for generalized thermodynamic systems with memory. For the systems that simulate viscoelasticity the solutions of a dissipation inequality characterize either the work to be done to bring a viscoelastic body from the reference to the given state, or the work that can be done by the body on its passage from the given to the reference state [5].

Thus, of interest is the problem of describing the set of solutions of a dissipation inequality for linear passive and relaxation dynamic systems. For second-order relaxation systems this problem was solved in [5]. In [9], for arbitrary-order relaxation systems convex polyhedrons ω_- and ω_+ were constructed that approximate from the inside and outside, respectively, the convex set ω which is one-to-one connected to the set of all the solutions of the dissipation inequality. In the present work, we give an algebraic-geometric description of the set of solutions of a dissipation inequality for third-order relaxation systems.

Basic Notions and Definitions. $X := \mathbf{R}^n$ and $U := \mathbf{R}^r$ are the Euclidean spaces of dimensionality n and r, respectively; $\langle \cdot,\cdot \rangle_X$ is the scalar product in the space X; $L(U, X)$ is the set of linear operators acting from U into X; $L(X) := L(X, X)$; (A, B, C) is the triple of operators, where $A \in L(X)$, $B \in L(U, X)$, and $C \in L(X, U)$; B^* is the operator conjugate to B (it is determined from the relation $\langle Bu, x \rangle_X = \langle u, B^* x \rangle_U \ \forall\ x \in X,\ \forall\ u \in U$) .

Let $Q \in L(X)$. The inequality $Q > 0$ means that $Q = Q^*$ and the quadratic form of $\langle Qx, x \rangle_X$ is strictly positive ($\langle Qx, x \rangle_X > 0\ \ \forall\ x \in X, x \neq 0$). If $Q \geq 0$, then $\langle Qx, x \rangle_X \geq 0\ \ \forall\ x \in X$. The condition that $Q_1 \geq Q_2$ is equivalent to the inequality $Q_1 - Q_2 \geq 0$.

The triple (A, B, C) is identified with the following dynamic system prescribed in the space of states:

Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, Belarus. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 73, No. 5, pp. 1033-1041, September–October, 2000. Original article submitted January 5, 2000.

 1062-0125/00/7305-1010$25.00

$$\Sigma : \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad t \in \mathbf{R},$$

where $x \in X$, $u \in U$, and $y \in U$.

If the condition

$$\int_{-\infty}^{t} \langle u(\tau), y(\tau) \rangle_U d\tau \geq 0, \quad \forall t \in \mathbf{R}, \quad \forall u(\cdot) \in H,$$

is satisfied, where H is the set of locally summed up and integrated (with a square) functions with carriers restricted on the left, then the system Σ is called passive.

The results of [10] yield that the following statements are equivalent: 1) the system Σ is passive; 2) there is the solution $Q > 0$ of the inequality

$$LQ := A^* Q + QA \leq 0, \tag{1}$$

that satisfies the condition

$$QB = C^*; \tag{2}$$

3) if the pair $(x(t), u(t))$, $t \in \mathbf{R}$, satisfies the system Σ, then the dissipation inequality is satisfied:

$$\frac{1}{2} \frac{d}{dt} \langle Qx(t), x(t) \rangle_X \leq \langle u(t), y(t) \rangle_U, \quad \forall t \in \mathbf{R}, \quad \forall u(\cdot) \in H. \tag{3}$$

The set Ω of all the solutions of system (1) and (2) coincides with the set of the solutions of dissipation inequality (3), is convex and compact in $L(X)$, and also has maximum, $Q^+ > 0$, and minimum, $Q^- > 0$, elements, so that $Q^- \leq Q \leq Q^+$ holds for any solution Q [10]. According to [5], the solutions $Q^\pm$ determine the minimum and maximum thermodynamic potentials for linear thermodynamic systems with memory.

We determine the boundary $\partial\Omega$ of the convex set Ω. Let $\tilde{\Omega}$ be the relative interior of Ω, i.e., the totality of internal points of the set Ω, if Ω is considered as a subset of its affine shell. Then $\partial\Omega := \Omega \setminus \tilde{\Omega}$.

Next we consider the problem of describing the set Ω of the solutions of dissipation inequality (3) for a subclass of passive systems, namely, third-order relaxation systems:

$$\Sigma = (A, B, C), A = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3), B = C^* = (b_1, b_2, b_3)^*, \tag{4}$$

where $\lambda_3 < \lambda_2 < \lambda_1 < 0$; $b_i > 0$, $i = 1, 2, 3$.

The solution of this problem will require several standard notions from algebraic geometry. An algebraic set or variety is the name given to a subset of the space $\mathbf{R}^3$ consisting of all the joint real zeros of a finite number of polynomials of three variables with coefficients from the field of real numbers. The geometric image of nontrivial varieties in $\mathbf{R}^3$ is the points, lines, and surfaces of the dimensionality zero, one, and two respectively.

For the variety P ($\dim P = 2$) prescribed by the equation $F(x_1, x_2, x_3) = 0$ the point $(x_1^*, x_2^*, x_3^*) \in P$ is called the singular point if $\frac{\partial F}{\partial x_i}(x_1^*, x_2^*, x_3^*) = 0$, $\forall\, i = 1, 2, 3$. The dimensionality of the variety tangent to P is equal to two at a nonsingular point and to three at the singular one [11].

Definition 1. *The part P_0 ($\dim P_0 = 2$) of the variety P will be called a limited convex component of the variety P if P_0 is the boundary of the convex limited set in $\mathbf{R}^3$.*

We note that the limited convex component P_0 represents the surface in $\mathbf{R}^3$; however, P_0 need not be a variety in the sense specified above.

Algebraic-Geometric Parametrization of the Set Ω. First, we will determine the set ω in $\mathbf{R}^3$ connected to the set Ω in a one-to-one manner. We will consider the system (1) and (2) for the matrix $Q = (q_{ij})$, $i, j = 1, 2, 3$, $(q_{ij} = q_{ji})$. From Eq. (2) it follows that

$$q_{11} = 1 - m_{12}\, q_{12} - m_{13}\, q_{13}\,, \quad q_{22} = 1 - m_{12}^{-1} q_{12} - m_{23}\, q_{23}\,,$$

$$q_{33} = 1 - m_{13}^{-1} q_{13} - m_{23}^{-1} q_{23}\,, \tag{5}$$

where $m_{ij} = b_j b_i^{-1} > 0$, $(i, j) \in \gamma := \{(i, j) : 1 \le i < j \le 3\}$.

We assume that $q = (q_{12}, q_{13}, q_{23}) \in \mathbf{R}^3$. According to (5), the matrix Q from (2) depends on the vector q, with the condition $Q(q) \in \Omega$ being satisfied if and only if $q \in \{q : LQ(q) \le 0\} = :\omega$. Consequently, the mapping $q \to Q(q)$ $(\forall\ q \in \omega)$ assigns a one-to-one correspondence between ω and Ω, with the set ω being a convex compact and the boundary ∂ω corresponding to the boundary ∂Ω. Hence it follows that the problem of the description of a set of solutions of dissipation inequality (3) is equivalent to the problem of description of ω, which, owing to the convexity of ω, is reduced to finding the boundary ∂ω.

Theorem 1. *The surface ∂ω coincides with the sole limited convex component of the variety*

$$P = \{q : \det LQ\,(q) = 0\}\,. \tag{6}$$

The proof of Theorem 1 and of subsequent statements are given in the Supplement.

We now pass to the problem of calculation of the maximum Q^+ and minimum Q^- solutions of dissipation inequality (3). We will determine the points q^+, $q^- \in \omega$ by the equality $Q(q^{\pm}) = Q^{\pm}$. The following is true:

Theorem 2. *The points q^+ and q^- are singular for the variety P.*

We present a system of equations for the singular points of the variety P belonging to ∂ω. For this we introduce the following notation:

$$n_{ij} := \lambda_j \lambda_i^{-1}\,, \quad \alpha_{ij} := \frac{n_{ij} + n_{ij}^{-1}}{2}\,, \quad \gamma_{ij} = \frac{1 + \alpha_{12} + \alpha_{13} + \alpha_{23}}{2\,(1 + \alpha_{ij})}\,, \quad (i, j) \in \gamma\,. \tag{7}$$

Theorem 3. *The point $q = (q_{12}, q_{13}, q_{23}) \in \partial\omega$ is singular if and only if it is a nontrivial solution of the system of equations*

$$\begin{aligned} &\gamma_{23} q_{12} q_{13} + m_{12} q_{12} q_{23} + m_{13} q_{13} q_{23} = q_{23}\,, \\ &m_{12}^{-1} q_{12} q_{13} + \gamma_{13} q_{12} q_{23} + m_{23} q_{13} q_{23} = q_{13}\,, \\ &m_{13}^{-1} q_{12} q_{13} + m_{23}^{-1} q_{12} q_{23} + \gamma_{12} q_{13} q_{23} = q_{12}\,. \end{aligned} \tag{8}$$

From each of the equations of system (8) we will express one of the variables, e.g., q_{12}:

$$q_{12} = q_{23} \frac{1 - m_{13} q_{13}}{\gamma_{23} q_{13} + m_{12} q_{23}} = q_{13} \frac{1 - m_{23} q_{23}}{m_{12}^{-1} q_{13} + \gamma_{13} q_{23}} = \frac{\gamma_{12} q_{13} q_{23}}{1 - m_{13}^{-1} q_{13} - m_{23}^{-1} q_{23}}\,.$$

Since $q \ne (0, 0, 0)$, to find q_{13} and q_{23} we have the system

$$(1 - m_{13} q_{13})\,(1 - m_{13}^{-1} q_{13} - m_{23}^{-1} q_{23}) = \gamma_{12} q_{13}\,(\gamma_{23} q_{13} + m_{12} q_{23})\,,$$

$$(1 - m_{23} q_{23})\,(1 - m_{13}^{-1} q_{13} - m_{23}^{-1} q_{23}) = \gamma_{12} q_{23}\,(\gamma_{13} q_{23} + m_{12}^{-1} q_{13})$$

or

TABLE 1. Singular Points of the Variety P for the System Σ_{example}

Coordinate axes	q^-	q^1	q^2	q^+
$x := q_{12}$	0.1316	−0.1966	0.2542	−0.3799
$y := q_{13}$	0.0452	−0.0569	−0.0878	0.1105
$z := q_{23}$	0.3322	0.4101	−1.2310	−1.5196

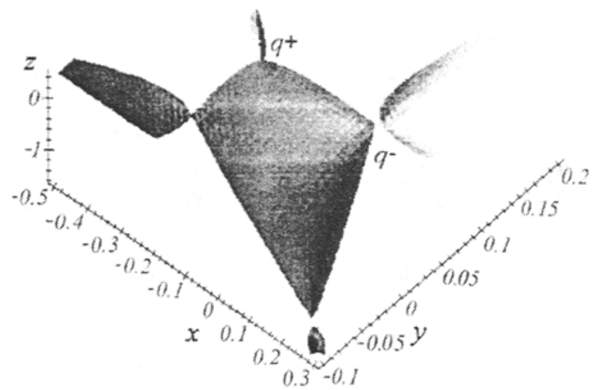

Fig. 1. Surface $\partial\omega$ as a part of the variety P for the system Σ_{example}.

$$q_{13}^2(\gamma_{12}\gamma_{23}-1)+q_{13}q_{23}m_{12}(\gamma_{12}-1)+q_{13}(m_{13}+m_{13}^{-1})+q_{23}m_{23}^{-1}-1=0\,,$$

$$q_{23}^2(\gamma_{12}\gamma_{13}-1)+q_{13}q_{23}m_{12}^{-1}(\gamma_{12}-1)+q_{23}(m_{23}+m_{23}^{-1})+q_{13}m_{13}^{-1}-1=0\,. \qquad (9)$$

Equations (9) prescribe two conics that, as can easily be verified, have a hyperbolic type. Thus, from Theorem 3 follows

Corollary 1. The singular points of the variety P that belong to $\partial\omega$ are determined by the points of the intersection of two hyperbolic-type conics prescribed by Eqs. (9).

Theorem 4. *1) The number of singular points of the variety P that lie on $\partial\omega$ is equal to four;*

2) *the coordinates of the singular points satisfy the inequality $q_{12}q_{13}q_{23} > 0$, with all four singular points being located on the surface $\partial\omega$ in different octants;*

3) *for the points $q^{\pm}$ the inequalities $q_{12}^- > 0$, $q_{13}^- > 0$, $q_{23}^- > 0$; $q_{12}^+ < 0$, $q_{13}^+ > 0$, $q_{23}^+ < 0$ hold.*

On the basis of Theorems 1-4 it is possible to give the following qualitative picture of the surface $\partial\omega$. It is known [11] that if there are two singular points on the variety of the third degree, then the straight line passing through them lies entirely on this variety. In our case, $\partial\omega$ is the limited convex component of a cubic variety and, consequently, six segments lie on $\partial\omega$ that connect four singular points. Figuratively speaking, our figure represents an "inflated" tetrahedron.

We will illustrate the obtained results in relation to the relaxation system Σ_{example}: $(\lambda_1, \lambda_2, \lambda_3, b_1, b_2, b_3) = (-1, -100, -1000, 1, 1, 1)$. The singular points of the variety P for it are given in Table 1 accurate to four decimal places. The surface $\partial\omega$ as a part of the variety P is indicated in Fig. 1.

Supplement. *Proof of Theorem 1.* According to the Sylvester criterion [12], condition (1) for system (4) in the symbols of (7) takes on the form

$$q_{11} \geq 0\,,\ q_{22} \geq 0\,,\ q_{33} \geq 0\,,$$

$$\frac{1}{2}W_{12} := 2q_{11}q_{22} - (1+\alpha_{12})\,q_{12}^2 \geq 0\,, \qquad (S1)$$

$$\frac{1}{2} W_{13} := 2q_{11}q_{33} - (1+\alpha_{13})\, q_{13}^2 \geq 0\,,$$

$$\frac{1}{2} W_{23} := 2q_{22}q_{33} - (1+\alpha_{23})\, q_{23}^2 \geq 0\,,$$

$$\frac{1}{4} W_{123} := 2q_{11}q_{22}q_{33} + (1+\alpha_{12}+\alpha_{13}+\alpha_{23})\, q_{12}q_{13}q_{23} -$$

$$-(1+\alpha_{12})\, q_{12}^2 q_{33} - (1+\alpha_{13})\, q_{13}^2 q_{22} - (1+\alpha_{23})\, q_{23}^2 q_{11} \geq 0\,.$$

Investigating simultaneously the inequalities $W_{12} \geq 0$ and $q_{11} \geq 0$ (or $W_{12} \geq 0$ and $q_{22} \geq 0$), we see that they separate one-half (with its interior) of the cone $W_{12} = 0$. Similarly, studying the remaining inequalities of the first and second degree, we come to the conclusion that the first six inequalities in system (S1) determine the intersection of three half-cones with their interiors.

Next, we consider the intersection of the cone $W_{12} = 0$ with a set of points assigned by the inequality $W_{123} \geq 0$. We have

$$(1+\alpha_{12}+\alpha_{13}+\alpha_{23})\, q_{12}q_{13}q_{23} - (1+\alpha_{13})\, q_{13}^2 q_{22} - (1+\alpha_{23})\, q_{23}^2 q_{11} \geq 0\,,$$

$$2q_{11}q_{22} = (1+\alpha_{12})\, q_{12}^2\,.$$

In the system obtained the inequality will additionally be multiplied by $2q_{22}$ and, using the equality, we obtain

$$2\,(1+\alpha_{12}+\alpha_{13}+\alpha_{23})\, q_{12}q_{13}q_{23}q_{22} - 2\,(1+\alpha_{13})\, q_{13}^2 q_{22}^2 - (1+\alpha_{12}) \times$$

$$\times (1+\alpha_{23})\, q_{12}^2 q_{23}^2 \geq 0\,. \tag{S2}$$

It can easily be seen that for the coefficients α_{ij}, $(i, j) \in \gamma$, the following identity holds:

$$(1+\alpha_{12}+\alpha_{13}+\alpha_{23})^2 = 2\,(1+\alpha_{12})\,(1+\alpha_{13})\,(1+\alpha_{23})\,. \tag{S3}$$

With account for (S3), inequality (S2) takes on the form

$$((1+\alpha_{12}+\alpha_{13}+\alpha_{23})\, q_{12}q_{23} - 2\,(1+\alpha_{13})\, q_{13}q_{22})^2 \leq 0\,. \tag{S4}$$

Relation (S4) can be satisfied only when it degenerates to an equality.

Thus, the cone $W_{12} = 0$ refers to the third-degree variety defined by the equation $W_{123} = 0$. The same can be said about the cones $W_{13} = 0$ and $W_{23} = 0$.

Further we note that the planes $q_{11} = 0$, $q_{22} = 0$, and $q_{33} = 0$ do not intersect and do not touch the surface $\partial\omega$, since their equations are not compatible with the conditions $W_{12} \geq 0$, $W_{13} \geq 0$, and $W_{23} \geq 0$.

We obtained that $\partial\omega$ coincides with that part of the variety P which is contained within the cones determined by the inequalities $W_{ij}(q) \geq 0$, $(i, j) \in \gamma$. Since ω is a convex compact, $\partial\omega$ is the limited convex component of the variety P. It is evident that the cubic variety cannot contain several limited convex components. Theorem 1 is proved.

Proof of Theorem 2. The existence of Q^+ can be written as $\forall\ x \in \mathbf{R}^3\ \exists\ \operatorname{extr}_{q \in \partial\omega} \langle x, Q(q)x \rangle = \langle x, Q^{\pm}x \rangle$. The operators $Q^{\pm}$ are independent of x and, consequently, an extremum can be sought at any x. Having taken sequentially three single unit vectors as x, we come to the conclusion that

$$\arg \min_{q \in \partial\omega} \langle x, Q(q)x \rangle = \arg \min_{q \in \partial\omega} q_{ii}(q)\,,$$

$$\arg\max_{q\in\partial\omega}\langle x, Q(q)x\rangle = \arg\max_{q\in\partial\omega} q_{ii}(q)\,,\ i=1,2,3\,.$$

Hence it follows that the dimensionalities of tangent varieties at extremal points are equal to three. Thus, the points sought are located among the singular points of the variety P. Theorem 2 is proved.

Proof of Theorem 3. The singular points of the variety P satisfy the system $W_{123}(q) = 0$ and $\nabla W_{123}(q) = 0$. In expanded form it is written so (the argument q will be omitted for brevity):

$$W_{123} = m_{12}^{-1}W_{13} + m_{12}W_{23} + W_3 = m_{13}^{-1}W_{12} + m_{13}W_{23} + W_2 =$$

$$= m_{23}^{-1}W_{12} + m_{23}W_{13} + W_1 = 0\,. \tag{S5}$$

Here the following designations are introduced:

$$W_1 = W_{12,13} + W_{13,12} = 4\,(1+\alpha_{23})\,q_{23}q_{11} - 2\,(1+\alpha_{12}+\alpha_{13}+\alpha_{23})\,q_{12}q_{13}\,,$$

$$W_2 = -\,(W_{12,23} + W_{23,12}) = 4\,(1+\alpha_{13})\,q_{13}q_{22} - 2\,(1+\alpha_{12}+\alpha_{13}+\alpha_{23})\,q_{12}q_{23}\,,$$

$$W_3 = W_{13,23} + W_{23,13} = 4\,(1+\alpha_{12})\,q_{12}q_{33} - 2\,(1+\alpha_{12}+\alpha_{13}+\alpha_{23})\,q_{13}q_{23}\,,$$

$$W_{12,13} = 2\,(1+n_{23}^{-1})\,q_{11}q_{23} - (1+n_{12})\,(1+n_{13}^{-1})\,q_{12}q_{13}\,,$$

where

$$W_{13,12} = 2\,(1+n_{23})\,q_{11}q_{23} - (1+n_{12}^{-1})\,(1+n_{13})\,q_{12}q_{13}\,,$$

$$W_{12,23} = -\,2\,(1+n_{13}^{-1})\,q_{22}q_{13} + (1+n_{12}^{-1})\,(1+n_{23}^{-1})\,q_{12}q_{23}\,,$$

$$W_{23,12} = -\,2\,(1+n_{13})\,q_{22}q_{13} + (1+n_{12})\,(1+n_{23})\,q_{12}q_{23}\,,$$

$$W_{13,23} = 2\,(1+n_{12}^{-1})\,q_{33}q_{12} - (1+n_{23})\,(1+n_{13}^{-1})\,q_{13}q_{23}\,,$$

$$W_{23,13} = 2\,(1+n_{12})\,q_{33}q_{12} - (1+n_{13})\,(1+n_{23}^{-1})\,q_{13}q_{23}\,.$$

Lemma 1. *On $\partial\omega$ system (S5) is equivalent to the system*

$$W_{12} = W_{13} = W_{23} = W_1 = W_2 = W_3 = 0\,. \tag{S6}$$

Proof. In system (S5) we expand the determinant W_{123} in the first column $W_{123} = 2q_{11}W_{23} - (1 + n_{12})(q_{12}W_{13,23} + (1+n_{13})q_{13}W_{12,23}$ and over the first line $W_{123} = 2q_{11}W_{23} - (1+n_{12}^{-1})q_{12}W_{23,13} + (1+n_{13}^{-1})q_{13}W_{23,12}$. Adding up, we will obtain $2W_{123} = 4q_{11}W_{23} - q_{12}((1+n_{12})W_{13,23} + (1+n_{12}^{-1})W_{23,13}) + q_{13}((1+n_{13})W_{12,23} + (1+n_{13}^{-1})W_{23,12}) = 4q_{11}W_{23} - 2q_{12}W_3 - 2q_{13}W_2$.

Expansions over the second line and second column have the form $W_{123} = 2q_{22}W_{13} - (1+n_{12})q_{12}W_{13,23} - (1+n_{23}^{-1})q_{23}W_{13,12}$, $W_{123} = 2q_{22}W_{13} - (1+n_{12}^{-1})q_{12}W_{23,13} + (1+n_{23})q_{23}W_{12,13}$. Their sum gives $2W_{123} = 4q_{22}W_{13} - q_{12}((1+n_{12})W_{13,23} + (1+n_{12}^{-1})W_{23,13}) - q_{23}((1+n_{23})W_{12,13} + (1+n_{23}^{-1})W_{13,12}) = 4q_{22}W_{13} - 2q_{12}W_3 - 2q_{23}W_1$.

Finally, expanding the determinant W_{123} over the remaining line and column, we will have

$$W_{123} = 2q_{33}W_{12} + (1+n_{13})\,q_{13}W_{12,23} - (1+n_{23})\,q_{23}W_{12,13}\,,$$

$$W_{123} = 2q_{33}W_{12} + (1 + n_{13}^{-1})\, q_{13}W_{23,12} - (1 + n_{23}^{-1})\, q_{23}W_{13,12}\,,$$

$$2W_{123} = 4q_{33}W_{12} + q_{13}\,((1 + n_{13})\, W_{12,23} + (1 + n_{13}^{-1})\, W_{23,12}) -$$

$$- q_{23}\,((1 + n_{23})\, W_{12,13} + (1 + n_{23}^{-1})\, W_{13,12}) = 4q_{33}W_{12} - 2q_{13}W_2 - 2q_{23}W_1\,.$$

As a result, system (S5) takes on the form

$$\begin{aligned} &m_{12}^{-1}W_{13} + m_{12}W_{23} + W_3 = 0\,,\\ &m_{13}^{-1}W_{12} + m_{13}W_{23} + W_2 = 0\,,\\ &m_{23}^{-1}W_{12} + m_{23}W_{13} + W_1 = 0\,,\\ &2q_{11}W_{23} - q_{12}W_3 - q_{13}W_2 = 0\,,\\ &2q_{22}W_{13} - q_{12}W_3 - q_{23}W_1 = 0\,,\\ &2q_{33}W_{12} - q_{13}W_2 - q_{23}W_1 = 0\,. \end{aligned} \tag{S7}$$

We will consider it as a linear system of six equations for six unknown quantities W_{12}, W_{13}, W_{23}, W_1, W_2, and W_3. Having eliminated W_1, W_2, and W_3, we arrive at the system

$$\begin{aligned} &m_{13}^{-1}q_{13}W_{12} + m_{12}^{-1}q_{12}W_{13} + (1 + q_{11})\, W_{23} = 0\,,\\ &m_{23}^{-1}q_{23}W_{12} + (1 + q_{22})\, W_{13} + m_{12}q_{12}W_{23} = 0\,,\\ &(1 + q_{33})\, W_{12} + m_{23}q_{23}W_{13} + m_{13}q_{13}W_{23} = 0\,. \end{aligned} \tag{S8}$$

The determinant Δ of this system is equal to

$$\Delta = -(q_{11}q_{22}q_{33} - q_{11}q_{23}^2 - q_{22}q_{13}^2 - q_{33}q_{12}^2 + 2q_{12}q_{13}q_{23}) -$$

$$-(q_{11}q_{22} - q_{12}^2) - (q_{11}q_{33} - q_{13}^2) - (q_{22}q_{33} - q_{23}^2) - q_{11} - q_{22} - q_{33} - 1\,.$$

We note that in parentheses are the principal minors of the matrix Q. Since on $\partial\omega$ the matrix Q is defined positively, $\Delta < 0$. Consequently, the sole solution of system (S8) is zero. In other words, system (S5) is equivalent to system (S6) on $\partial\omega$. Lemma 1 is proved.

Lemma 2. 1) $W_{12} = W_{13} = W_{23} = W_{123} = 0 \Rightarrow W_1^2 = W_2^2 = W_3^2 = 0$; 2) *at* $q_{ij} \neq 0$, $(i, j) \in \gamma$, *the reverse is true*: $W_1 = W_2 = W_3 = 0 \Rightarrow W_{12} = W_{13} = W_{23} = W_{123} = 0$.

Proof. 1) We will show, for example, the validity of the equality $W_2^2 = 0$; the remaining two are derived by analogy.

From $W_{123} = 0$ it follows that

$$2q_{11}q_{22}q_{33} + (1 + \alpha_{12} + \alpha_{13} + \alpha_{23})\, q_{12}q_{13}q_{23} -$$

$$- (1 + \alpha_{12})\, q_{12}^2q_{33} - (1 + \alpha_{13})\, q_{13}^2q_{22} - (1 + \alpha_{23})\, q_{23}^2q_{11} = 0\,. \tag{S9}$$

Taking into account that $W_{12} = 0$, from (S9) we obtain $(1 + \alpha_{12} + \alpha_{13} + \alpha_{23})q_{12}q_{13}q_{23} - (1 + \alpha_{13})q_{13}^2q_{22} - (1 + \alpha_{23})q_{23}^2q_{11} = 0$. We additionally multiply the latter relation by $2q_{22}$ and again apply the equality $W_{12} = 0$:

$$2(1+\alpha_{12}+\alpha_{13}+\alpha_{23})\,q_{12}q_{13}q_{23}q_{22}-2(1+\alpha_{13})\,q_{13}^2q_{22}^2-(1+\alpha_{12})\times$$

$$\times(1+\alpha_{23})\,q_{12}^2q_{23}^2=0$$

or $((1+\alpha_{12}+\alpha_{13}+\alpha_{23})q_{12}q_{23}-2(1+\alpha_{13})q_{13}q_{22})^2=0$, i.e., $W_2^2=0$.

2) Since $W_2=0$, we have a chain of equalities:

$$q_{13}W_{12}=4q_{11}q_{22}q_{13}-2(1+\alpha_{12})\,q_{12}^2q_{13}=2\frac{1+\alpha_{12}+\alpha_{13}+\alpha_{23}}{(1+\alpha_{13})}\,q_{11}q_{12}q_{23}-$$

$$-2(1+\alpha_{12})\,q_{12}^2q_{13}=\frac{1+\alpha_{12}}{1+\alpha_{12}+\alpha_{13}+\alpha_{23}}\,q_{12}\,(4(1+\alpha_{23})\,q_{11}q_{23}-$$

$$-2(1+\alpha_{12}+\alpha_{13}+\alpha_{23})\,q_{12}q_{13})=\frac{1+\alpha_{12}}{1+\alpha_{12}+\alpha_{13}+\alpha_{23}}\,q_{12}W_1\,.$$

We see that at $q_{ij}\neq 0$, $(i,j)\in\gamma$, the equality $W_1=0$ entails $W_{12}=0$. Similarly we can also show that $W_{13}=W_{23}=0$. The expansion of $W_{123}=2q_{11}W_{23}-q_{12}W_3-q_{13}W_2=0$ completes the proof of Lemma 2.

On the basis of Lemmas 1 and 2 we can conclude that each nontrivial solution of the system

$$W_1=W_2=W_3=0 \tag{S10}$$

is the solution of system (S5) and, vice versa, any solution of system (S5) on $\partial\omega$ satisfies system (S10). In the designations of (7) system (S10) will be written as

$$q_{11}q_{23}-\gamma_{23}q_{12}q_{13}=q_{22}q_{13}-\gamma_{13}q_{12}q_{23}=q_{33}q_{12}-\gamma_{12}q_{13}q_{23}=0\,. \tag{S11}$$

With account for equalities (5) Eqs. (S11) will take the form of (8).

If one of the nondiagonal elements q_{ij}, $(i,j)\in\gamma$, is equal to zero, then, according to (8), the remaining ones will also be equal to zero, but the point $q=0$ does not lie on $\partial\omega$. Thus, we can state that the singular points of the variety P belonging to $\partial\omega$ are the nontrivial solutions of the system of equations (8) and they alone. Theorem 3 is proved.

Proof of Theorem 4. First we note that a set of relaxation systems is bound in $\mathbf{R}^3\times\mathbf{R}^3$, since the space of the parameters of such systems represents the Cartesian product of two cones: $V_1=\{(\lambda_1,\lambda_2,\lambda_3):\lambda_3<\lambda_2<\lambda_1<0\}$ and $V_2=\{b_i:b_i>0,\ i=1,2,3\}$. Continuously changing the parameters $(\lambda_1,\lambda_2,\lambda_3,b_1,b_2,b_3)\in V_1\times V_2$, it is possible to obtain any other system not leaving the class of relaxation systems. In this case the coefficients m_{ij}, γ_{ij}, $(i,j)\in\gamma$, may be considered as continuous functions of λ_i, b_i, $i=1,2,3$, in the domain of their determination $V_1\times V_2$.

For the values of the parameters of the system Σ_{example} the statements of Theorem 4 are valid. We will prove that with change in the parameters $(\lambda_1,\lambda_2,\lambda_3,b_1,b_2,b_3)\in V_1\times V_2$ the quantity of the singular points of the variety P that lie on $\partial\omega$, i.e., of nontrivial real solutions of system (8), remains a constant number equal to four.

System (9) can be reduced to the fourth-degree equation

$$a_4q_{23}^4+\ldots+a_0=0\,. \tag{S12}$$

The coefficient a_4 of this equation is equal to $a_4=(\gamma_{12}\gamma_{13}-1)(\gamma_{12}\gamma_{13}-1)(\gamma_{12}\gamma_{23}-1)-(\gamma_{12}-1)^2)$. We will represent the factors in terms of α_{ij}, $(i,j)\in\gamma$, using identity (S3) and estimates $\alpha_{ij}>1$, $(i,j)\in\gamma$, that directly follow from the inequalities $\lambda_3<\lambda_2<\lambda_1<0$:

$$\gamma_{12}\gamma_{13} - 1 = \frac{(1 + \alpha_{12} + \alpha_{13} + \alpha_{23})^2}{4(1 + \alpha_{12})(1 + \alpha_{13})} - 1 = \frac{1 + \alpha_{23}}{2} - 1 = \frac{\alpha_{23} - 1}{2} > 0 ;$$

$$(\gamma_{12}\gamma_{13} - 1)(\gamma_{12}\gamma_{23} - 1) - (\gamma_{12} - 1)^2 = \frac{\alpha_{23} - 1}{2}\,\frac{\alpha_{13} - 1}{2} - \frac{(\alpha_{13} + \alpha_{23} - \alpha_{12} - 1)^2}{4(1 + \alpha_{12})^2} =$$

$$= \frac{(\alpha_{13} - 1)(\alpha_{23} - 1)}{4} - \frac{(\alpha_{13} - 1)(\alpha_{23} - 1)}{2(1 + \alpha_{12})} = \frac{(\alpha_{12} - 1)(\alpha_{13} - 1)(\alpha_{23} - 1)}{4(1 + \alpha_{12})} > 0 .$$

Both factors are strictly positive; therefore $a_4 \neq 0$, and Eq. (S12) cannot degenerate into an equation of a smaller degree. Consequently, Eq. (S12) always has four roots, and we are to show that with change in the parameters of the system the roots will remain real.

From the theorem of continuous dependence of the roots of an algebraic equation on its coefficients [13] and from the reality of the coefficients in (S12), it follows that Eq. (S12) will not have complex roots if it does not have multiple roots at any values of the parameters $(\lambda_1, \lambda_2, \lambda_3, b_1, b_2, b_3) \in V_1 \times V_2$. The appearance of the multiple root is possible only on convergence of two singular points. For this purpose they must intersect a certain coordinate plane, since the singular points of the variety P of the initial system Σ_{example} are located in different octants. However, this is not essential, since if $q_{ij} = 0$ for some $(i, j) \in \gamma$, then, according to Eq. (8), $q = (0, 0, 0)$, but the coordinate origin does not lie on $\partial\omega$. Thus, the points do not leave their octants. For the singular points of the variety P of the system Σ_{example} the inequalities $q_{12}^- > 0$, $q_{13}^- > 0$, $q_{23}^- > 0$; $q_{12}^+ < 0$, $q_{13}^+ > 0$, $q_{23}^+ < 0$; $q_{12}q_{13}q_{23} > 0$ satisfied. From what has been said above and from the singularity of $q^{\pm}$ it follows that they remain valid for the singular points belonging to $\partial\omega$ and with change in the parameters $(\lambda_1, \lambda_2, \lambda_3, b_1, b_2, b_3) \in V_1 \times V_2$. The theorem is proved.

REFERENCES

1. W. A. Day, *The Thermodynamics of Simple Materials with Fading Memory*, Springer–Verlag (1972).
2. N. Petrov and I. Brankov, *Contemporary Problems of Thermodynamics* [in Russian], Moscow (1986).
3. V. L. Kolpashchikov and A. I. Shnip, *Inzh.-Fiz. Zh.*, **68**, No. 4, 633-646 (1995).
4. V. L. Kolpashchikov and A. I. Shnip, *Inzh.-Fiz. Zh.*, **68**, No. 5, 724-737 (1995).
5. V. T. Borukhov, V. L. Kolpashchikov, and A. I. Shnip, *Inzh.-Fiz. Zh.*, **69**, No. 5, 707-715 (1996).
6. V. A. Yakubovich, *Dokl. Akad. Nauk SSSR*, **143**, No. 6, 1304-1307 (1962).
7. R. E. Kalman, *Proc. Natl. Acad. Sci. USA*, **309**, No. 5, 327-357 (1963).
8. C. Truesdell, *A First Course in Rational Continuum Mechanics*, Baltimore, Maryland (1972).
9. V. T. Borukhov, *Differents. Uravn.*, **35**, No. 1, 15-23 (1999).
10. J. Willems, *J. Franklin Inst.*, **301**, No. 6, 605-621 (1978).
11. I. R. Shafarevich, *Fundamentals of Algebraic Geometry* [in Russian], Vol. 1, Moscow (1988).
12. F. R. Gantmacher, *Theory of Matrices* [Russian translation], Moscow (1988).
13. A. M. Ostrovskii, *Solution of Equations and Systems of Equations* [in Russian], Moscow (1963).